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Nonvacuum Taub-Type Cosmological Model

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The Einstein universe is a simple model describing a static cosmological spacetime, having a constant radius and a constant curvature, and, as is well known, it does not describe our universe. We propose a model which is an extension of Einstein's. Our metric, having $R \times S^3$ topology, describes a nonisotropic homogeneous closed (finite) universe of Bianchi type IX. This metric is similar to that of Taub, but is simpler. Unlike the Taub solution (which is a cosmological extension of the NUT solution), however, the universe described by our metric contains matter. Like the Taub metric, our metric has two positive constants (τ, T). The gravitational red shift calculated from our metric is given. Similarly to the Schwarzschild metric, which has a "singularity" at $r = 2m$, this metric has the same kind of "singularity" at $t = 2\tau$. The maximal extension of the coordinates in our metric is fairly analogous to that of the Schwarzschild metric.

1. INTRODUCTION

It is well known that the simplest cosmological model is the Einstein universe, which is static and nonempty, having a constant radius and a constant curvature. It is of type IX in the classification of Bianchi (Weinberg, 1972). The three Friedmann–Robertson–Walker cosmological models describe an isotropic homogeneous universe. One of these is a closed (finite) expanding (and contracting) universe of Bianchi type IX, and is a generalization of the Einstein universe (Landau and Lifshitz, 1975).

Another model of Bianchi type IX is the familiar Taub metric, which is spatially homogeneous, but not isotropic (Taub, 1951). It satisfies the vacuum Einstein field equations and has no singularity; thus, its physical applicability is very limited. One can find an interesting analogy between the gravitational spacetime described by the NUT metric (Newman *et al.*, 1963) and the cosmological model of Taub. The Taub–NUT metric has two positive constants (m, l), and the NUT metric reduces to the Schwarzschild metric for $l = 0$, All the above models have $R \times S^3$ topology.

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Recently, much interest has focused on models which are nonisotropic (MacCallum, 1979, 1984), especially in connection with the early universe, as well as with Misner's (1969, 1970) mixmaster universes. In this paper we propose such a model. Our metric has $R \times S^3$ topology (Bianchi type IX) (Ozsvàth and Schücking, 1969; Ryan and Shepley, 1975) and, like the Taub metric, has two positive constants (τ, T), though it is much simpler than the latter. It is a solution of Einstein's field equations with an energy-momentum tensor describing matter (the same source of the Einstein universe), along with a stress tensor. In a way, one can think of our metric as the cosmological analogy to the Schwarzschild solution, especially concerning the singularity problems at $t=0$ and $t=2\tau$ in our metric, as compared to $r=0$ and $r=2m$ for the Schwarzschild case (Carmeli, 1982).

In Section 2 we present our metric and discuss its properties. Use will be made of standard tetrad methods of differential forms. In Section 3 we discuss the gravitational red shift problem related to our metric. The maximal extension of the metric is then given in Section 4, whereas Section 5 is devoted to the summary and concluding remarks.

2. THE METRIC

In this section our cosmological metric will be introduced in the coordinate system x^μ (Greek letters take the values 0, 1, 2, 3): $x^0 = t$, $x^1 = \theta$, $x^2 = \phi$, $x^3 = \psi$. Here, θ, ϕ, ψ are Euler angles, where $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$, and $0 \leq \psi \leq 4\pi$. We will also use the tetrad of differential forms ω^α, where $\omega^0 = dt$ and the other components ω^a (Latin indices take the values 1, 2, 3) are expressed in terms of the Euler angles by the well-known relations familiar from classical mechanics for the rigid-body rotations, when use is made of the body axes (Goldstein, 1965)

$$\begin{aligned} \omega^1 &= \sin\theta \sin\psi \, d\phi + \cos\psi \, d\theta \\ \omega^2 &= \sin\theta \cos\psi \, d\phi - \sin\psi \, d\theta \\ \omega^3 &= \cos\theta \, d\phi + d\psi \end{aligned} \tag{2.1}$$

Our discussion will start with the extension of the Einstein static universe, which is of Bianchi type IX. The Einstein field equations which yield this model are

$$G^{\mu\nu} = \rho_0 u^\mu u^\nu \tag{2.2}$$

Here and throughout this paper we will use units in which $c = G = \kappa = 1$. In equation (2.2) ρ_0 is the matter density, a positive constant, u^μ is the four-velocity, $u^\mu = dx^\mu/ds$, and one has $u^\mu = 0$ for $\mu \neq 0$. Here $G^{\mu\nu}$ is the Einstein tensor. Our extension introduces the field equations in the form

$$G^{\mu\nu} = \rho_0 u^\mu u^\nu + P^{\mu\nu} \tag{2.3}$$

where $P^{\mu\nu}$ is a stress tensor, and u^{μ} satisfies the same conditions outlined above.

Our goal is to obtain a homogeneous nonisotropic cosmological model, and for simplicity we will assume that the anisotropy occurs in only one direction. We will choose this distinct direction to be along ω^3, while the other two directions, along ω^1 and ω^2, are supposed to be isotropic in our tetrad system. Due to the above requirements of symmetries, the metric we are seeking will be assumed to have the following simple form:

$$ds^2 = V^{-1}\,dt^2 - t^2[(\omega^1)^2 + (\omega^2)^2] - T^2 V(\omega^3)^2 \tag{2.4}$$

Here T is a positive constant which has the dimensions of time, V is a function of time only, $V = V(t)$, and the signature is $(+---)$. Substituting (2.1) in the metric (2.4), the latter will have the form

$$ds^2 = V^{-1}\,dt^2 - t^2(d\theta^2 + \sin^2\theta\,d\phi^2) - T^2 V(d\psi + \cos\theta\,d\phi)^2 \tag{2.5}$$

in the coordinate system $x^{\mu} = (t, \theta, \phi, \psi)$. It should be noted that the metric (2.5) is not a particular case of the Taub metric given by

$$ds^2 = U^{-1}\,dt^2 - (t^2 + l^2)(d\theta^2 + \sin^2\theta\,d\phi^2) - 4l^2 U(d\psi + \cos\theta\,d\phi)^2 \tag{2.6}$$

From the metric (2.5) one can calculate the Christoffel symbols, the Ricci tensor, the Ricci scalar, and the Einstein tensor. In the coordinate system $x^{\mu} = (t, \theta, \phi, \psi)$, the contravariant Einstein tensor is then given by (all other quantities are given in Appendix A)

$$\begin{aligned}
G^{00} &= Vt^{-1}\dot{V} + V^2 t^{-2}\left(1 - \frac{T^2}{4t^2}\right) + Vt^{-2} \\
G^{01} &= G^{02} = G^{03} = 0 \\
G^{11} &= -\frac{\ddot{V}}{2t^2} - t^{-3}\dot{V} - \frac{T^2 V}{4t^6} \\
G^{12} &= G^{13} = 0 \\
G^{22} &= G^{11}\sin^{-2}\theta \\
G^{23} &= -G^{22}\cos\theta \\
G^{33} &= G^{11}\cot^2\theta - T^{-2}V^{-1}t^{-1}\dot{V} - T^{-2}V^{-1}t^{-2}(1 + V) + \frac{3}{4t^4}
\end{aligned} \tag{2.7}$$

Here a dot denotes differentiation with respect to the time coordinate t.

In the tetrad system, the contravariant Einstein tensor can be easily calculated using standard tensor transformation rules. The resultant tensor is diagonal and is given by

$$G_T^{\mu\nu} = \begin{pmatrix} G^{00} & 0 & 0 & 0 \\ 0 & G^{11} & 0 & 0 \\ 0 & 0 & G^{11} & 0 \\ 0 & 0 & 0 & G^{33} - G^{11}\cot^2\theta \end{pmatrix} \tag{2.8}$$

(All other quantities are given in Appendix B.) Here G^{00}, G^{11}, and G^{33} are given explicitly in equations (2.7). The Einstein field equations in the tetrad system will have the matrix form

$$\begin{pmatrix} G^{00} & 0 & 0 & 0 \\ 0 & G^{11} & 0 & 0 \\ 0 & 0 & G^{11} & 0 \\ 0 & 0 & 0 & G^{33} - G^{11}\cot^2\theta \end{pmatrix} = \begin{pmatrix} \rho_0(u^0)^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \\ P^{\mu\nu} \\ \\ \end{pmatrix} \tag{2.9}$$

It can easily be seen from equation (2.9) that $P^{\mu\nu}$ must have a diagonal form, namely $P^{\mu\nu} = 0$ for $\mu \neq \nu$. The explicit Einstein field equations, according to equation (2.9), are given by

$$G_T^{00} = Vt^{-1}\dot{V} + V^2 t^{-2}\left(1 - \frac{T^2}{4t^2}\right) + Vt^{-2} + \rho + P_0 \tag{2.10a}$$

$$G_T^{11} = -\frac{\ddot{V}}{2t^2} - t^{-3}\dot{V} - \frac{T^2 V}{4t^6} = P_1 \tag{2.10b}$$

$$G_T^{22} = -\frac{\ddot{V}}{2t^2} - t^{-3}\dot{V} - \frac{T^2 V}{4t^6} = P_2 \tag{2.10c}$$

$$G_T^{33} = -T^{-2}V^{-1}t^{-1}\dot{V} - T^{-2}V^{-1}t^{-2}(1+V) + \frac{3}{4t^4} = P_3 \tag{2.10d}$$

where $\rho = \rho_0(u^0)^2$ and $P_0 = P^{00}$, $P_1 = P^{11}$, $P_2 = P^{22}$, $P_3 = P^{33}$. From equations (2.10b) and (2.10c) one notices that $P_1 = P_2$, expressing the isotropy along these two spatial directions.

Let us now define a new function

$$W = (V+1)t, \qquad V = -1 + W/t \tag{2.11}$$

Thus, we obtain

$$\dot{W} = t\dot{V} + V + 1 \tag{2.12a}$$

$$\ddot{W} = t\ddot{V} + 2\dot{V} \tag{2.12b}$$

Substituting (2.11) and (2.12) in (2.10), we obtain

$$G_T^{00}=t^{-2}\left(\frac{W}{t}-1\right)\dot{W}-\frac{T^2}{4t^4}\left(\frac{W}{t}-1\right)^2=\rho+P_o \tag{2.13a}$$

$$G_T^{11}=G_T^{22}=-2t^{-3}\ddot{W}-\frac{T^2}{4t^6}\left(\frac{W}{t}-1\right)=P_1=P_2 \tag{2.13b}$$

$$G_T^{33}=-\frac{\dot{W}}{T^2t^2(W/t-1)}+\frac{3}{4t^4}=P_3 \tag{2.13c}$$

A particular solution to the above equations is given by

$$W=\text{const} \tag{2.14}$$

Thus, we obtain the following relations:

$$\rho+P_0=-\frac{T^2V^2}{4t^4} \tag{2.15a}$$

$$P_1=P_2=-\frac{T^2V}{4t^6} \tag{2.15b}$$

$$P_3=\frac{3}{4t^4} \tag{2.15c}$$

with

$$V=-1+2\tau/t \tag{2.16}$$

where $\tau=W/2$ is a constant having the dimension of time and is assumed to be of the order of 10^{10} years.

At $t=0$ the universe has the shape of infinitely long, infinitesimally thin, string. For values of $t<2\tau$, V is positive; thus, the stress tensor has two identical negative components P_1 and P_2 and one positive component P_3. The negative ones describe expansions along the two directions ω^1 and ω^2, while the positive component expresses a contraction along the ω^3 direction. Such a stress tensor will then yield an anisotropic universe expanding along two directions and contracting along the third one, thus having the shape of a symmetrical ellipsoid. When t becomes larger than 2τ, $t>2\tau$, V and g_{00} become negative, while g_{33} becomes positive. Therefore, spacelike and timelike are exchanged, just as in the Schwarzschild metric at $r=2m$. A more detailed discussion on this singularity is given in Section 4.

The two components of the stress tensor P_1 and P_2 change signs at $t>2\tau$, while the component P_3 remains positive, which means contraction along all spatial directions. The contraction along the ω^3 direction continues, but at $t=2\tau$ the expansion occurring at $t<2\tau$ of the universe along the ω^1 and ω^2 directions ceases and contraction along these directions starts.

In the next section, the red shift analysis will be carried out.

3. THE RED SHIFT

When discussing the red shift problem, it is more convenient and meaningful to use the tetrad system (2.1) rather than the coordinates $x^\mu = (t, \theta, \phi, \psi)$. Our first calculation of the red shift will be along the ω^1 direction, which is identical to that along the ω^2 direction. The calculation is standard, and is as follows (Carmeli, 1982).

Consider two clocks moving along the ω^1 direction at two points denoted by 1 and 2. The line elements at these two points, using equation (2.4) and the fact that $dt = 0$, are given by

$$ds^2(1) = -t_1^2(\omega^1)^2 \tag{3.1a}$$

$$ds^2(2) = -t_2^2(\omega^1)^2 \tag{3.1b}$$

The ratio of the rates of similar clocks located at these points is given by

$$\frac{ds(2)}{ds(1)} = \frac{t_2}{t_1} \tag{3.2}$$

The frequency ν_0 of an atom located at point 1 at a cosmological time t_1, when measured by an observer located at point 2 at a cosmological time t_2 $(t_1 < t_2 < 2\tau)$, is therefore given by

$$\frac{\nu_0}{\nu} = \frac{t_2}{t_1} \tag{3.3}$$

Let us denote $t_2 - t_1 = \Delta t$; then from equation (3.3) one has

$$\frac{\nu_0}{\nu} = \frac{t_2}{t_2 - \Delta t} = \left(1 - \frac{\Delta t}{t_2}\right)^{-1}$$

and, thus, to a first approximation,

$$\frac{\nu_0}{\nu} \cong 1 + \frac{\Delta t}{t_2} > 1 \tag{3.4}$$

which means a red shift. Hence, our universe expands in the ω^1 (and in the ω^2) direction, in accordance with the results of the previous section.

The red shift parameter is given by $z = \nu_0/\nu - 1$. Using equation (3.4), we obtain

$$z = \Delta t / t_2 \tag{3.5}$$

Denoting $t_2 = t$, and defining $c\Delta t = R$, where R represents the distance between the observer and the source, we obtain

$$z = \frac{1}{c}\frac{R}{t} \tag{3.6}$$

Equation (3.6) is the well-known Hubble law $z = HR/c$; thus, the Hubble constant, according to the above analysis, is

$$H = 1/t \tag{3.7}$$

where t is the present age of the universe.

When use is made of the other form of the Hubble law, $v = HR$, where v is the speed of recession of the galaxies, one can easily see that when t is small, the speed of recession is large, in agreement with the well-known early inflationary model (Guth, 1981; Linde, 1982). As t becomes larger, v becomes smaller.

The red shift calculation along the ω^3 direction is obtained in the same way as is done along the ω^1 and ω^2 directions, though it is a bit complicated. The line element has the form $ds^2 = -T^2V(\omega^3)^2$; hence at the two points 1 and 2 we have

$$ds^2(1) = -T^2V(1)(\omega^3)^2 \tag{3.8a}$$

$$ds^2(2) = -T^2V(2)(\omega^3)^2 \tag{3.8b}$$

The ratio of the frequencies, using the value of V given by (2.16), is

$$\frac{\nu_0}{\nu} = \left(-1+\frac{2\tau}{t_2}\right)^{1/2}\left(-1+\frac{2\tau}{t_1}\right)^{-1/2} = \left(\frac{t_1}{t_2}\right)^{1/2}\left(\frac{2\tau-t_2}{2\tau-t_1}\right)^{1/2} \tag{3.9}$$

For values of $t_1 < t_2 < 2\tau$, both terms on the right-hand side of equation (3.9) are smaller than 1; thus, $\nu_0 < \nu$, which means that, instead of a red shift, one finds a blue shift.

This result is consistent with the contraction along the ω^3 direction shown in the previous section; the galaxies move toward each other. Writing equation (3.9) in the form

$$\frac{\nu_0}{\nu} = \left(\frac{t_1}{t_2}\right)^{1/2}\left(1-\frac{t_2}{2\tau}\right)^{1/2}\left(1-\frac{t_1}{2\tau}\right)^{-1/2} \tag{3.10}$$

then for values $t_1 < t_2 \ll 2\tau$ one can use Taylor's expansion, getting

$$\begin{aligned}\frac{\nu_0}{\nu} &= \left(\frac{t_1}{t_2}\right)^{1/2}\left(1-\frac{t_2}{4\tau}\right)\left(1+\frac{t_1}{4\tau}\right)\\ &\cong \left(\frac{t_1}{t_2}\right)^{1/2}\left[1+\frac{1}{4\tau}(t_1-t_2)\right]\end{aligned} \tag{3.11}$$

to a first approximation. Let us denote $t_2 - t_1 = \Delta t$; thus,

$$\left(\frac{t_1}{t_2}\right)^{1/2} = \left(\frac{t_2-\Delta t}{t_2}\right)^{1/2} \cong 1-\frac{\Delta t}{2t_2} \qquad (\Delta t \ll t_2)$$

and equation (3.11) will have the form

$$\frac{\nu_0}{\nu} \cong \left(1-\frac{\Delta t}{2t}\right)\left(1-\frac{\Delta t}{4\tau}\right) = 1-\Delta t\left(\frac{1}{2t}+\frac{1}{4\tau}\right) < 1 \tag{3.12}$$

to a first approximation, and where $t = t_2$. Using the above equation, the red shift parameter is given by

$$z = -\frac{1}{2}\left(\frac{1}{2\tau} + \frac{1}{t}\right)\Delta t \qquad (3.13)$$

Again defining $c\,\Delta t = R$, we obtain

$$z = -\frac{1}{2c}\left(\frac{1}{2\tau} + \frac{1}{t}\right)R \qquad (3.14)$$

which is the Hubble law. The Hubble constant, according to the above calculation, is

$$H = \frac{1}{2}\left(\frac{1}{2\tau} + \frac{1}{t}\right) \qquad (3.15)$$

In the next section a maximal extension of the coordinate system $x^\mu = (t, \theta, \phi, \psi)$ is introduced and discussed.

4. MAXIMAL EXTENSION OF THE METRIC

In the previous sections one can notice the interesting properties of the singularity at $t = 2\tau$. We now investigate this singularity. The analysis is similar to that for the Schwarzschild metric (see, for instance, Carmeli, 1977, 1982).

We start our discussion by finding the null directions, in the coordinate system $x^\mu = (t, \theta, \phi, \psi)$, along which $d\theta = d\phi = 0$, given by

$$ds^2 = V^{-1}\,dt^2 - T^2 V\,d\psi^2 = 0 \qquad (4.1)$$

Hence the null directions satisfy the equation

$$\frac{dt}{d\psi} = \pm TV \qquad (4.2)$$

Substituting the value of V given by equation (2.16), we obtain

$$\frac{dt}{d\psi} = \pm\frac{T}{t}(2\tau - t) \qquad (4.3)$$

In the region $t < 2\tau$, the opening of the light cone has an angle π at $t = 0$, and decreases to zero as t increases to 2τ. On the other hand, in the region $t > 2\tau$, the parametric lines of the coordinate t become spacelike and those of the coordinate ψ become timelike. Consequently, the light cones rotate 90° (see Figure 1), and their openings decrease from π (at $t \to 2\tau_+$) to $\pm T$ (at $t \to \infty$).

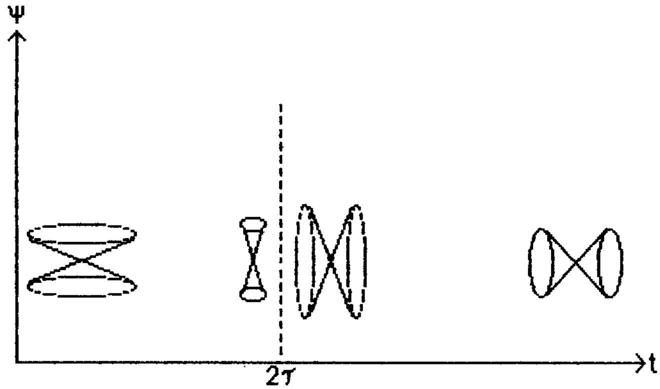

Fig. 1. Orientations and openings of the light cones.

Comparing now the two different figures of the light cones on both sides of $t=2\tau$, we see that the regions on the two sides of the universal surface $t=2\tau$ do not join smoothly there.

At the cosmological time $t=2\tau$ the components of the metric (2.5) g_{00} diverge and g_{33}, g_{23} vanish, thus giving the impression of the existence of a singularity. The determinant of the metric (2.5), $g=-T^2t^4\sin^2\theta$, however, is regular at $t=2\tau$, while the Ricci scalar, given in Appendix A [equation (A.10)] vanishes.

It follows that the singularity at $t=2\tau$ is not an intrinsic feature of the metric (2.5) and is different from that at $t=0$, but rather is a property of the coordinate system used in expressing the metric.

We now introduce a new coordinate system, $x'^{\mu}=(u, \theta, \phi, v)$, in which light rays everywhere have the slope $dx'^3/dx'^0=\pm 1$. The transformation of coordinates is as follows:

$$\left.\begin{aligned} u&=\left(1-\frac{t}{2\tau}\right)^{1/2} e^{t/4\tau}\cosh\left(T\frac{\psi}{4\tau}\right)\\ v&=\left(1-\frac{t}{2\tau}\right)^{1/2} e^{t/4\tau}\sinh\left(T\frac{\psi}{4\tau}\right)\end{aligned}\right\}\quad (t<2\tau) \tag{4.4}$$

$$\left.\begin{aligned} u&=\left(\frac{t}{2\tau}-1\right)^{1/2} e^{t/4\tau}\sinh\left(T\frac{\psi}{4\tau}\right)\\ v&=\left(\frac{t}{2\tau}-1\right)^{1/2} e^{t/4\tau}\cosh\left(T\frac{\psi}{4\tau}\right)\end{aligned}\right\}\quad (t>2\tau) \tag{4.5}$$

The inverse transformation is given by

$$u^2-v^2=\left(1-\frac{t}{2\tau}\right)e^{t/2\tau} \tag{4.6}$$

both for $t<2\tau$ and $t>2\tau$, and

$$\psi=(4\tau/T)\,\mathrm{tgh}^{-1}(v/u)\qquad (t<2\tau) \tag{4.7a}$$

$$\psi=(4\tau/T)\,\mathrm{tgh}^{-1}(u/v)\qquad (t>2\tau) \tag{4.7b}$$

It will be noticed that u is the timelike coordinate, while v is the spacelike one; both u and v are dimensionless.

The metric tensor $g'_{\mu\nu}$ in the new coordinate system has the form

$$g'_{\mu\nu}=\begin{pmatrix} f^2 & 0 & 0 & 0\\ 0 & -t^2 & 0 & 0\\ 0 & 0 & -(t^2\sin^2\theta+T^2V\cos^2\theta) & g'_{23}\\ 0 & 0 & g'_{32} & -f^2\end{pmatrix} \tag{4.8}$$

where the function f depends on t alone and is defined by

$$f^2=\frac{32\tau^3}{t}\,e^{-t/2\tau}$$

$$=\text{a transcendental function of } u^2-v^2 \tag{4.9}$$

and the components $g'_{23}=g'_{32}$ have the form

$$g'_{23}=\frac{4\sqrt{2}T}{t}\,\tau^{3/2}\,e^{-t/4\tau}\cos\theta\begin{cases}-(2\tau-t)^{1/2}\left[\cosh\left(T\dfrac{\psi}{4\tau}\right)\right]^{-1} & (t<3\tau)\\ (t-2\tau)^{1/2}\left[\sinh\left(T\dfrac{\psi}{4\tau}\right)\right]^{-1} & (t>2\tau)\end{cases} \tag{4.10}$$

The metric (4.8) does not have a singularity at $t=2\tau$; the only singularity in this metric is at $t=0$.

The new coordinates give an analytic extension S' of the limited region of spacetime S which is described without singularity by the coordinate system $x^\mu=(t, \theta, \phi, \psi)$ with $t<2\tau$. The metric in the extended region joins on smoothly, and without singularity, to the metric at the boundary of the region S at $t=2\tau$ (see Figure 2). The extended region S', moreover, is the maximal possible singularity-free extension of the region S. The properties indicated in Figure 2 are a direct consequence of equations (4.6) and (4.7). The region I in Figure 2 corresponds to $0\le t<2\tau$. An object moving from the hyperbola $t=0$ will cross the straight lines $\psi=\text{const}$ in the direction of increasing ψ, which means expansion. On the other hand, in region II ($t>2\tau$), the same object will cross the straight lines ($\psi=\text{const}$) in the direction of decreasing ψ, which means contraction.

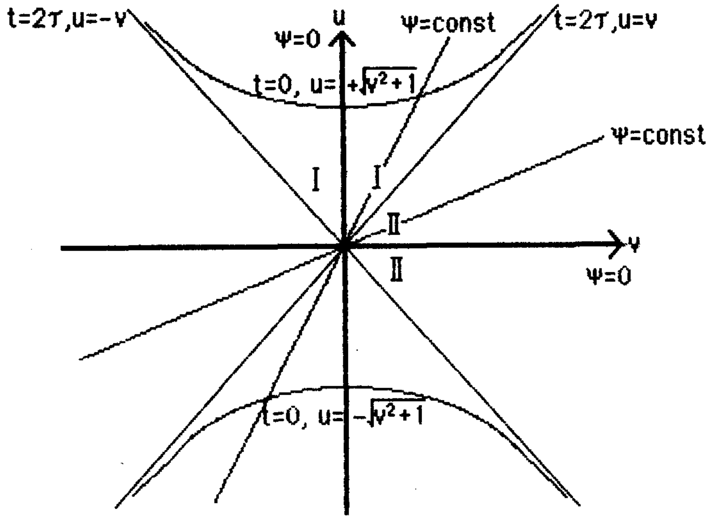

Fig. 2. Maximal extension of the metric in the u, v plane.

5. SUMMARY AND CONCLUDING REMARKS

In this paper we introduced and discussed a new homogeneous and nonisotropic cosmological metric describing a closed and nonstatic universe. The most interesting property of this universe is its development in time.

At the very beginning, our universe has the shape of an infinitesimal thin string with an infinite length, and in time it evolves into the shape of a symmetrical ellipsoid. This result is due to an expansion in two spatial directions, and a contraction in the third one.

At the cosmological time $t = 2\tau$, there exists a "singularity," which was discussed in detail in Section 4. At this time the expansion along the two spatial directions ceases and contraction begins, while along the other direction the contraction continues; thus, the whole universe contracts toward a singularity.

As is well known, at the present time the universe is isotropic and homogeneous (Carmeli *et al.*, 1981, 1983, 1984). Red shift measurements, interpreted as a Doppler effect, indicate an expanding universe. Accordingly, our metric cannot describe our universe at the present time. It suggests, however, a new model for the early inflationary universe.

APPENDIX A

In this appendix, for the convenience of the reader, we give the expressions for the Christoffel symbols, the Ricci tensor, the Ricci scalar, and the Einstein tensor in the coordinates $x^\alpha = (t, \theta, \phi, \psi)$, where θ, ϕ, ψ are the Euler angles, $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$, $0 \le \psi \le 4\pi$.

The metric is given by

$$g_{\mu\nu} = \begin{pmatrix} V^{-1} & 0 & 0 & 0 \\ 0 & -t^2 & 0 & 0 \\ 0 & 0 & -(t^2 \sin^2\theta + T^2 V \cos^2\theta) & -T^2 V \cos\theta \\ 0 & 0 & -T^2 V \cos\theta & -T^2 V \end{pmatrix} \tag{A.1}$$

$$g^{\mu\nu} = \begin{pmatrix} V & 0 & 0 & 0 \\ 0 & -t^{-2} & 0 & 0 \\ 0 & 0 & -t^{-2}\sin^{-2}\theta & t^{-2}\cos\theta\sin^{-2}\theta \\ 0 & 0 & t^{-2}\cos\theta\sin^{-2}\theta & -(T^{-2}V^{-1} + t^{-2}\cot^2\theta) \end{pmatrix} \tag{A.2}$$

$$g = \det g_{\mu\nu} = -T^2 t^4 \sin^2\theta, \qquad \sqrt{-g} = T t^2 \sin\theta \tag{A.3}$$

where V is a function of t only and T is a constant having a dimension of time, presumably of the order of 10^{10} years. Notice the existence of the following simple relationships among the metric components, which show that everything, except for g^{00} $(= g_{00}V^2)$, can be expressed in terms of g_{11} and g_{33}:

$$g_{22} = g_{11}\sin^2\theta + g_{33}\cos^2\theta, \qquad g_{23} = g_{33}\cos\theta \tag{A.4}$$

$$g^{11} = g_{11}t^{-4}, \qquad g^{22} = g_{11}t^{-4}\sin^{-2}\theta, \qquad g^{23} = -g_{11}t^{-4}\cos\theta\sin^{-2}\theta$$
$$g^{33} = g_{11}t^{-4}\cot^2\theta + g_{33}T^{-4}V^{-2} \tag{A.5}$$

The nonvanishing Christoffel symbols are given by

$$\Gamma^0_{00} = -\tfrac{1}{2}V^{-1}\dot V, \qquad \Gamma^0_{11} = Vt, \qquad \Gamma^0_{22} = V(t\sin^2\theta + \tfrac{1}{2}T^2\dot V\cos^2\theta)$$

$$\Gamma^0_{23} = \tfrac{1}{2}T^2 V\dot V\cos\theta, \qquad \Gamma^0_{33} = \tfrac{1}{2}T^2 V\dot V, \qquad \Gamma^1_{01} = t^{-1}$$

$$\Gamma^1_{22} = \left(\frac{T^2 V}{t^2} - 1\right)\sin\theta\cos\theta, \qquad \Gamma^1_{23} = \frac{T^2 V}{2t^2}\sin\theta, \qquad \Gamma^2_{02} = t^{-1} \tag{A.6}$$

$$\Gamma^2_{12} = \left(1 - \frac{T^2 V}{2t^2}\right)\cot\theta, \qquad \Gamma^2_{13} = -\frac{T^2 V}{2t^2}\sin^{-1}\theta$$

$$\Gamma^3_{02} = (\tfrac{1}{2}V^{-1}\dot V - t^{-1})\cos\theta, \qquad \Gamma^3_{03} = \tfrac{1}{2}V^{-1}\dot V$$

$$\Gamma^3_{12} = \left(\frac{T^2 V}{2t^2} - 1\right)\cos\theta\cot\theta - \tfrac{1}{2}\sin\theta, \qquad \Gamma^3_{13} = \frac{T^2 V}{2t^2}\cot\theta$$

In the above and following formulas a dot denotes differentiation with respect to the time coordinate t.

We use the following expression for the Ricci tensor (Carmeli, 1982):

$$R_{\mu\nu}=\frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\,\Gamma^{\alpha}_{\mu\nu})}{\partial x^{\alpha}}-\frac{\partial^2\ln\sqrt{-g}}{\partial x^{\mu}\partial x^{\nu}}-\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} \tag{A.7}$$

We then have

$$\begin{aligned}
&R_{00}=-V^{-1}(\tfrac{1}{2}\ddot{V}+t^{-1}\dot{V}), \qquad R_{01}=R_{02}=R_{03}=0\\
&R_{11}=t\dot{V}+\left(1-\frac{T^2}{2t^2}\right)V+1, \qquad R_{12}=R_{13}=0\\
&R_{22}=R_{11}\sin^2\theta+R_{33}\cos^2\theta, \qquad R_{23}=R_{33}\cos\theta\\
&R_{33}=\tfrac{1}{2}T^2V(\ddot{V}+2t^{-1}\dot{V}+T^2t^{-4}V)
\end{aligned} \tag{A.8}$$

The contravariant components are given by

$$\begin{aligned}
&R^{00}=R_{00}V^2, \qquad R^{01}=R^{02}=R^{03}=0\\
&R^{11}=R_{11}t^{-4}, \qquad R^{12}=R^{13}=0\\
&R^{22}=R_{11}t^{-4}\sin^{-2}\theta, \qquad R^{23}=-R^{22}\cos\theta=-R_{11}t^{-4}\cos\theta\sin^{-2}\theta\\
&R^{33}=R_{11}t^{-4}\cot^2\theta+R_{33}T^{-4}V^{-2}
\end{aligned} \tag{A.9}$$

The Ricci scalar is given by

$$R=-\left\{\ddot{V}+4t^{-1}\dot{V}+\frac{2}{t^2}\left[\left(1-\frac{T^2}{4t^2}\right)V+1\right]\right\} \tag{A.10}$$

The covariant components of the Einstein tensor are given by

$$\begin{aligned}
&G_{00}=t^{-1}V^{-1}\dot{V}+t^{-2}\left(1-\frac{T^2}{4t^2}\right)+t^{-2}V^{-1}, \qquad G_{01}=G_{02}=G_{03}=0\\
&G_{11}=-\tfrac{1}{2}t^2\ddot{V}-t\dot{V}-\frac{T^2V}{4t^2}, \qquad G_{12}=G_{13}=0\\
&G_{22}=G_{11}\sin^2\theta+G_{33}\cos^2\theta, \qquad G_{23}=G_{33}\cos\theta\\
&G_{33}=t^{-1}T^2V\dot{V}-t^{-2}T^2V(1+V)+\frac{3}{4}\frac{T^4V^2}{t^4}
\end{aligned} \tag{A.11}$$

The contravariant components of the Einstein tensor are given by

$$\begin{aligned}
&G^{00}=G_{00}V^2, \qquad G^{01}=G^{02}=G^{03}=0\\
&G^{11}=G_{11}t^{-4}, \qquad G^{12}=G^{13}=0\\
&G^{22}=G_{11}t^{-4}\sin^{-2}\theta, \qquad G^{23}=-G_{11}t^{-4}\cos\theta\sin^{-2}\theta\\
&G^{33}=G_{11}t^{-4}\cot^2\theta+G_{33}T^{-4}V^{-2}
\end{aligned} \tag{A.12}$$

It will be noted that the dependence among the various components of the Ricci and Einstein tensors is identical to that among the components of the metric tensor as given by equations (A.4) and (A.5).

APPENDIX B

This appendix is devoted to results expressed in the tetrad (2.1):

$$\begin{aligned}\omega^1 &= \sin\theta \sin\psi \, d\phi + \cos\psi \, d\theta \\ \omega^2 &= \sin\theta \cos\psi \, d\phi - \sin\psi \, d\theta \\ \omega^3 &= \cos\theta \, d\phi + d\psi\end{aligned} \tag{B.1}$$

Expressions for the field quantities in this tetrad system are obtained, as usual, by coordinate transformations. Denoting $dx'^{\alpha} = (dt, \omega^1, \omega^2, \omega^3)$ and $x^{\beta} = (t, \theta, \phi, \psi)$, we obtain

$$\frac{\partial x'^{\alpha}}{\partial x^{\beta}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\psi & \sin\theta \sin\psi & 0 \\ 0 & -\sin\psi & \sin\theta\cos\psi & 0 \\ 0 & 0 & \cos\theta & 1 \end{pmatrix} \tag{B.2}$$

The metric, which is diagonal, has the form

$$g^T_{\mu\nu} = \begin{pmatrix} V^{-1} & 0 & 0 & 0 \\ 0 & -t^2 & 0 & 0 \\ 0 & 0 & -t^2 & 0 \\ 0 & 0 & 0 & -T^2 V \end{pmatrix} = \begin{pmatrix} g_{00} & 0 & 0 & 0 \\ 0 & g_{11} & 0 & 0 \\ 0 & 0 & g_{11} & 0 \\ 0 & 0 & 0 & g_{33} \end{pmatrix} \tag{B.3}$$

$$g^{\mu\nu}_T = \begin{pmatrix} V & 0 & 0 & 0 \\ 0 & -t^{-2} & 0 & 0 \\ 0 & 0 & -t^{-2} & 0 \\ 0 & 0 & 0 & -T^{-2}V^{-1} \end{pmatrix} = \begin{pmatrix} g^{00} & 0 & 0 & 0 \\ 0 & g^{11} & 0 & 0 \\ 0 & 0 & g^{11} & 0 \\ 0 & 0 & 0 & T^{-4}V^{-2}g_{33} \end{pmatrix} \tag{B.4}$$

$$g = \det g^T_{\mu\nu} = -T^2 t^4, \qquad \sqrt{-g} = Tt^2 \tag{B.5}$$

where g_{00}, g_{11}, g^{00}, and g^{11} are the components of the metric in the coordinate system $x^{\mu} = (t, \theta, \phi, \psi)$, given explicitly in Appendix A.

The Ricci tensor is given, for instance, by

$$R^{\mu\nu}_T = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} R^{\alpha\beta} \tag{B.6}$$

We obtain

$$R_T^{\mu\nu} = \begin{pmatrix} R^{00} & 0 & 0 & 0 \\ 0 & R^{11} & 0 & 0 \\ 0 & 0 & R^{11} & 0 \\ 0 & 0 & 0 & R_{33}T^{-4}V^{-2} \end{pmatrix} \tag{B.7}$$

$$R_{\mu\nu}^T = \begin{pmatrix} R_{00} & 0 & 0 & 0 \\ 0 & R_{11} & 0 & 0 \\ 0 & 0 & R_{11} & 0 \\ 0 & 0 & 0 & R_{33} \end{pmatrix} \tag{B.8}$$

$$R_T = R \tag{B.9}$$

where the expressions on the right-hand side of (B.7)-(B.9) are given explicitly in Appendix A.

For the Einstein tensor we obtain

$$G_T^{\mu\nu} = \begin{pmatrix} G^{00} & 0 & 0 & 0 \\ 0 & G^{11} & 0 & 0 \\ 0 & 0 & G^{11} & 0 \\ 0 & 0 & 0 & G_{33}T^{-4}V^{-2} \end{pmatrix} \tag{B.10}$$

$$G_{\mu\nu}^T = \begin{pmatrix} G_{00} & 0 & 0 & 0 \\ 0 & G_{11} & 0 & 0 \\ 0 & 0 & G_{11} & 0 \\ 0 & 0 & 0 & G_{33} \end{pmatrix} \tag{B.11}$$

where the expressions on the right-hand side of the above equations are given explicitly in Appendix A.

Notice that the same dependence exists between the components $g_{\mu\nu}^T$, $R_{\mu\nu}^T$ and $G_{\mu\nu}^T$.

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